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# INTEGRABLE SYSTEMS IN STRINGY GRAVITY

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## Abstract

Static axisymmetric Einstein–Maxwell–Dilaton and stationary axisymmetric Einstein–Maxwell–Dilaton–Axion theories in four space–time dimensions are shown to be integrable by means of the inverse scattering transform method.

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Both vacuum and electrovacuum Einstein equations enjoy a complete integrability property being restricted to space-times admitting a two-parameter Abelian group of isometries [1]. This entails rich mathematical structures such as an infinite set of non-local conservation laws [2] and Backlund transformations [3]. Similar integrability property is shared by more general gravity coupled systems including scalar and vector fields, which follow from certain Kaluza-Klein (KK) models [4]. New generalizations of Einstein equations arise in the zero-slope limit of the heterotic string theory, in four dimensions they include vector fields, a dilaton, an axion, and moduli fields [5]. It was shown recently that a *pure* gravity coupled to dilaton and axion is also 2-dim integrable [6]. However, the most intriguing features of string-motivated gravity, related to the black hole puzzle, are due to peculiar nature of the dilaton coupling to *vector* fields [7]. Here we show that two stringy gravity models including vector fields are also 2-dim integrable: static axisymmetric Einstein-Maxwell-Dilaton (EMD) system with an arbitrary dilaton coupling constant, and stationary axisymmetric Einstein-Maxwell-Dilaton-Axion (EMDA) system.

Our reasoning is based on the sigma-model approach used earlier to prove integrability of 2-dim reductions of vacuum and electrovacuum Einstein equations [8]. It consists in a derivation of the 3-dim sigma-model from 4-dim theory in a space-time possessing a Killing vector field, and a subsequent identification of the *symmetric space* structure of the target space. This implies possibility of *zero-curvature* representation of the equations of motion and applicability of the inverse scattering transform method [9], when the second Killing symmetry is imposed. The procedure is rather well-known, so we just outline its main steps and fix our notation.

Consider a general 4-dim coupled system of gravitational,  $U(1)$  vector, and some scalar massless fields. Assuming the metric to admit a time-like Killing symmetry, one can write the interval as

$$ds^2 = -f(dt - \omega_i dx^i)^2 + f^{-1} h_{ij} dx^i dx^j, \quad (1)$$

where  $f$ ,  $\omega_i$ , and the 3-metric  $h_{ij}$  depend on the space coordinates  $x^i, i = 1, 2, 3$ , only. Then the  $U(1)$  field is fully describable in terms of electric and magnetic potentials  $v$ ,  $a$ . Usually, in conformity with the Einstein constraints, a twist potential  $\chi$  may be introduced to generate the rotation one-form  $\omega_i$ . Together with  $f$  and scalars, these variables may be interpreted as a set of scalar fields constituting a source for  $h_{ij}$ . If there is no scalar potentials in the initial 4-dim action, the theory will be equivalent to a 3-dim sigma model

$$S_\sigma = \frac{1}{2} \int \left( \mathcal{R} - \mathcal{G}_{AB}(\varphi) \partial_i \varphi^A \partial_j \varphi^B h^{ij} \right) \sqrt{h} d^3x, \quad (2)$$

where  $\mathcal{R}$  is the 3-dim scalar curvature,  $\varphi^A = (f, \chi, v, a, \text{scalar fields})$ ,  $A = 1, \dots, K$ , and  $\mathcal{G}_{AB}$  is the target space metric.

Suppose that the target space is a *symmetric* riemannian space  $G/H$  with  $N$ -parameter isometry group  $G$  acting transitively on it ( $H$  being an isotropy subgroup), generated by the set of  $N$  Killing vectors forming the Lie algebra of  $G$ ,  $[K_\mu, K_\nu] = C^\lambda_{\mu\nu} K_\lambda$ ,  $\mu, \nu, \lambda = 1, \dots, N$ .

Then the equations of motion for  $\varphi^A$  will be equivalent to the set of conservation laws for Noether currents

$$\partial_i(h^{ij}\sqrt{h}J_i^\mu) = 0, \quad J_i^\mu = \tau_A^\mu \frac{\partial\varphi^A}{\partial x^i}, \quad (3)$$

built using the corresponding Killing one-forms  $\tau^\mu = \eta^{\mu\nu} K_\nu^A \mathcal{G}_{AB} d\varphi^B$ , where  $\eta^{\mu\nu}$  is an inverse to the Killing–Cartan metric  $\eta_{\mu\nu} = k C^\alpha_{\mu\beta} C^\beta_{\nu\alpha}$ . With a proper choice of  $k$  these one-forms will satisfy Maurer–Cartan equation

$$d\tau^\mu + \frac{1}{2} C^\mu_{\alpha\beta} \tau^\alpha \wedge \tau^\beta = 0. \quad (4)$$

Let  $e_\mu$  denote some matrix representation of the Lie algebra of  $G$ ,  $[e_\mu, e_\nu] = C^\lambda_{\mu\nu} e_\lambda$ . Define the following matrix-valued connection one-form:  $\mathcal{A} = \mathcal{A}_B d\varphi^B = e_\mu \tau^\mu$ . In view of (4), the corresponding curvature vanishes,

$$\mathcal{F}_{BC} = \mathcal{A}_{C,B} - \mathcal{A}_{B,C} + [\mathcal{A}_B, \mathcal{A}_C] = 0, \quad (5)$$

and thus  $\mathcal{A}_B$  is a pure gauge

$$\mathcal{A}_B = -(\partial_B g)g^{-1}, \quad g \in G. \quad (6)$$

The pull-back of  $\mathcal{A}$  onto the configuration space  $x^i$  is equivalent to (3) and, hence, to the equations of motion of the sigma-model. In terms of  $g$  the Eqs. (3) read

$$d\{(\star dg)g^{-1}\} = 0, \quad (7)$$

where a star stands for a 3-dim Hodge dual.

Now impose an axial symmetry condition, representing the 3-metric in the Lewis–Papapetrou form:

$$h_{ij}dx^i dx^j = e^{2\gamma}(d\rho^2 + dz^2) + \rho^2 d\varphi^2. \quad (8)$$

Then (7) becomes equivalent to a modified chiral equation

$$(\rho g, \rho g^{-1})_{,\rho} + (\rho g, z g^{-1})_{,z} = 0, \quad (9)$$

and the corresponding Lax pair with a complex spectral parameter  $\lambda$  can be found:

$$D_1 \Psi = \frac{\rho U - \lambda V}{\rho^2 + \lambda^2} \Psi, \quad D_2 \Psi = \frac{\rho V + \lambda U}{\rho^2 + \lambda^2} \Psi. \quad (10)$$

Here  $V = \rho g, \rho g^{-1}$ ,  $U = \rho g, z g^{-1}$ ,  $\Psi$  is a matrix "wave function", and

$$D_1 = \partial_z - \frac{2\lambda^2}{\rho^2 + \lambda^2} \partial_\lambda, \quad D_2 = \partial_\rho + \frac{2\lambda\rho}{\rho^2 + \lambda^2} \partial_\lambda \quad (11)$$

are commuting operators; then (9) follows from the compatibility condition  $[D_1, D_2]\Psi = 0$ . This linearization is sufficient to establish a desired integrability property. An inverse scattering transform method [9] can be directly applied to (10) to generate multisoliton solutions, and an infinite-dimensional algebra of a Geroch–Kinnersley–Chitre (GKC) type can be derived.

Let us apply this formalism to EMD and EMDA systems. The first is described by the action

$$S = \frac{1}{16\pi} \int \left( R - 2(\partial\phi)^2 - e^{-2\alpha\phi} F^2 \right) \sqrt{-g} d^4x, \quad (12)$$

where  $\phi$  is the real scalar field (dilaton),  $F = dA$  is the Maxwell two-form,  $\alpha$  is the dilaton coupling constant. For  $\alpha = 0$ , (12) reduces to the Brans–Dicke–Maxwell (BDM) action in the Einstein frame (with the Brans–Dicke parameter  $\omega = -1$ ). For  $\alpha = \sqrt{3}$ , (12) is derivable from the 5-dim KK-theory.

In conformity with the Maxwell equations following from (12), electric and magnetic potentials can be introduced via

$$F_{i0} = \frac{1}{\sqrt{2}} \partial_i v, \quad F^{ij} = -\frac{f}{\sqrt{2}h} e^{2\alpha\phi} \epsilon^{ijk} \partial_k a, \quad (13)$$

while the twist potential  $\chi$  is defined through

$$\tau_i = \partial_i \chi + v \partial_i a - a \partial_i v, \quad \tau^i = -f^2 \epsilon^{ijk} \partial_j \omega_k / \sqrt{h}, \quad (14)$$

(3-dim indices are raised and lowered using  $h_{ij}$ ). The corresponding target space is five-dimensional ( $K = 5$ ), and

$$\mathcal{G} = \frac{1}{2f^2} \left( df^2 + (d\chi + vda - adv)^2 \right) + \frac{1}{f} (e^{-2\alpha\phi} dv^2 + e^{2\alpha\phi} da^2) + 2d\phi^2. \quad (15)$$

For  $\alpha = 0$  and  $\phi = \text{const}$  this metric reduces to one given by Neugebauer and Kramer for the EM system [8].

It turns out that, for the general stationary class of metrics (1), the target space (15) is a symmetric riemannian space only for  $\alpha = 0, \sqrt{3}$ , when it has the structure of cosets  $SU(2, 1)/S(U(2) \times U(1)) \times R$  and  $SL(3, R)/SO(3)$  respectively, corresponding to BDM and 5-dimensional KK theories. For  $\alpha \neq 0, \sqrt{3}$  the isometry group of (15) is only  $N = 5$  solvable subgroup of  $SL(3, R)$ . However, if an additional condition of *staticity* is imposed,  $\omega_i = 0$ , the (truncated) target space possess the symmetric space property for *arbitrary*  $\alpha$ .

In the static case it is consistent to consider electric and magnetic configurations separately. Both will be described by the same equations after reparametrization

$$\xi = (\alpha\phi - 1/2 \ln f)/\nu, \quad \eta = (\phi + \alpha/2 \ln f)/\nu, \quad (16)$$

for a magnetic case, and

$$\xi = -(\alpha\phi + 1/2 \ln f)/\nu, \quad \eta = (\phi - \alpha/2 \ln f)/\nu, \quad (17)$$

for an electric one, where  $\nu = (\alpha^2 + 1)/2$ . Denoting as  $u$  either magnetic ( $a$ ) or electric ( $v$ ) potentials respectively, one can write the line element of the truncated three-dimensional target space as  $dl_3^2 = d\eta^2 + dl_2^2$  where

$$dl_2^2 = d\xi^2 + e^{2\nu\xi} du^2 \quad (18)$$

Since  $\eta$  decouples, it is sufficient to deal only with this 2-dim space, which can easily be shown to represent a coset  $SL(2, R)/U(1)$ . Indeed, one can find three Killing vectors for (18):

$$K_1 = \partial_u, \quad K_2 = p\partial_u - \nu^{-1}u\partial_\xi, \quad K_3 = u\partial_u - \nu^{-1}\partial_\xi, \quad (19)$$

where  $p = (u^2 - \nu^{-2}e^{-2\nu\xi})/2$ , with the  $sl(2, R)$  structure constants  $C^3_{12} = C^2_{32} = C^1_{13} = 1$ . The corresponding Killing–Cartan one-forms, with the normalization  $k = (2\nu)^{-2}$ , will satisfy (4), and  $dl_2^2 = 1/2 \eta_{\mu\nu} \tau^\mu \otimes \tau^\nu$ , where  $\eta_{\mu\nu} = 2k \text{ diag}(1, 1, -1)$ . Choosing as  $e_\mu$  a  $2 \times 2$  representation of  $sl(2, R)$ , one can find from (6) the following matrix  $g \in SL(2, R)/U(1)$ :

$$g = \nu e^{\nu\xi} \sqrt{2} \begin{pmatrix} u^2 - p & -u/\sqrt{2} \\ -u/\sqrt{2} & 1 \end{pmatrix}. \quad (20)$$

Alternatively, in view of the isomorphism  $SL(2, R) \sim SO(2, 1)$ , a  $3 \times 3$  representation in terms of  $SO(2, 1)/SO(2)$  coset can be derived. In the axisymmetric case both can be used in the Lax pair (10).

For  $\alpha = 0$  ( $\nu = 1/2$ ) the above theory reduces to the corresponding representation for electrovacuum. Since the underlying algebraic structure is  $\alpha$ -independent, already this fact is sufficient to reveal integrability of the static axisymmetric EMD system with arbitrary  $\alpha$ . However, the integrability of electrovacuum in the *stationary* case is not shared by the arbitrary- $\alpha$  EMD system.

Remarkably, the EMDA theory turns out to be integrable in the *stationary* axisymmetric case too. The EMDA action in four dimensions reads

$$S = \frac{1}{16\pi} \int \left\{ R - 2\partial_\mu \phi \partial^\mu \phi - \frac{1}{2} e^{4\phi} \partial_\mu \kappa \partial^\mu \kappa - e^{-2\phi} F_{\mu\nu} F^{\mu\nu} - \kappa F_{\mu\nu} \tilde{F}^{\mu\nu} \right\} \sqrt{-g} d^4x, \quad (21)$$

where  $\tilde{F}^{\mu\nu} = \frac{1}{2} E^{\mu\nu\lambda\tau} F_{\lambda\tau}$ ,  $\kappa$  is an axion field. An electric potential is still introduced through the first of Eqs.(13), while for  $a$  one has

$$e^{-2\phi} F^{ij} + \kappa \tilde{F}^{ij} = -f \epsilon^{ijk} \partial_k a / \sqrt{2h}. \quad (22)$$

For a twist potential (14) remains valid. The target space now is 6-dimensional ( $K = 6$ ), and its metric reads

$$\mathcal{G} = \frac{1}{2} e^{-4\phi} \omega_\kappa^2 + 2d\phi^2 + \frac{1}{2} \left( \frac{df^2}{f^2} + f^2 \omega_\chi^2 \right) + f \left\{ e^{2\phi} \omega_v^2 + e^{-2\phi} \omega_a^2 \right\}, \quad (23)$$

where

$$\begin{aligned}\omega_\kappa &= e^{4\phi} d\kappa, \quad \omega_\chi = f^{-2}(d\chi + vda - adv), \\ \omega_v &= f^{-1}e^{-2\phi}dv, \quad \omega_a = f^{-1}e^{2\phi}(da - \kappa dv).\end{aligned}\tag{24}$$

Note, that the EMDA theory does not include the EMD one as a particular case. Indeed, setting  $\kappa = 0$  gives a constraint  $F\tilde{F} = 0$ . Similarly, the EMD theory does not contain the EM one: setting  $\phi = 0$  gives another constraint  $F^2 = 0$ .

As it was shown recently [10], the space (23) possess a  $N = 10$  isometry group consisting of scale, 3 gauge, 3 axion–dilaton duality, and 3 Ehlers–Harrison–type transformations, which unify  $T$  and  $S$  string dualities in 4–dim zero–slope heterotic string theory. Here we will show that the target space is a *symmetric* space which can be identified with the coset  $SO(3, 2)/(SO(3) \times SO(2))$ . Denoting generators of  $SO(3, 2)$  by pair indices  $ab, a < b$ , where  $a, b = 0, \theta, 1, 2, 3$  correspond to the invariant metric  $G_{ab} = \text{diag}(-1, -1, 1, 1, 1)$ , one has

$$[M_{ab}M_{cd}] = G_{bc}M_{ad} - G_{ac}M_{bd} + G_{ad}M_{bc} - G_{bd}M_{ac}.\tag{25}$$

The set of 10 one–form satisfying Maurer–Cartan equations with the structure constants  $C^{cd}_{ab\,ef}$  from (25) reads as follows. An abelian subalgebra of  $so(3, 2)$  corresponds to

$$-\tau^{01} = \omega_1 + \omega_f, \quad \tau^{\theta 2} = \omega_1 - \omega_f - 2\omega_2,\tag{26}$$

where

$$\omega_1 = \kappa\omega_\kappa - 2d\phi + a(v\omega_\chi + 2\omega_a), \quad \omega_f = f^{-1}df + \chi\omega_\chi, \quad \omega_2 = v\omega_v + \tilde{a}\omega_a.\tag{27}$$

Introduce a recurrent sequence

$$\begin{aligned}\omega_3 &= \kappa\omega_a - \omega_v, \quad \omega_4 = a\omega_\chi + \omega_3, \quad \omega_5 = v\omega_\chi + \omega_a, \\ \omega_6 &= d\kappa - \kappa^2\omega_\kappa + 4\kappa d\phi - a(\omega_4 + \omega_3), \quad \omega_7 = \omega_\kappa + v(\omega_a + \omega_5), \\ \omega_8 &= a\tau^{01} - v\omega_6 - \chi\omega_3 + a\omega_2 + da, \\ \omega_9 &= v\tau^{\theta 2} - a\omega_7 - \chi\omega_a + v\omega_2 + dv, \\ \omega &= a\omega_9 - v\omega_8 + \chi(\chi\omega_\chi - \omega_2 - 2\omega_f) + d\chi.\end{aligned}\tag{28}$$

Then the remaining set will read

$$\begin{aligned}2\tau^{0\theta} &= \omega + \omega_6 - \omega_7 - \omega_\chi, \quad 2\tau^{02} = \omega - \omega_6 - \omega_7 + \omega_\chi, \\ -2\tau^{\theta 1} &= \omega + \omega_6 + \omega_7 + \omega_\chi, \quad 2\tau^{12} = \omega - \omega_6 + \omega_7 - \omega_\chi, \\ -\tau^{03} &= \omega_5 + \omega_8, \quad \tau^{13} = \omega_5 - \omega_8, \quad \tau^{\theta 3} = \omega_4 - \omega_9, \quad -\tau^{23} = \omega_4 + \omega_9.\end{aligned}\tag{29}$$

In terms of  $\tau^{ab}$  one has  $\mathcal{G}_{AB} = 1/2 \eta_{ab\,cd} \tau_A^{ab} \tau_B^{cd}$ , where  $\eta_{ab\,cd} = 1/12 C^{gh}_{ab\,ef} C^{ef}_{cd\,gh}$ .

Now, using an adjoint representation of  $so(3, 2)$ , one can build  $5 \times 5$  connection one–form  $\mathcal{A}$  and the corresponding matrix  $g \in SO(3, 2)/(SO(3) \times SO(2))$ . Fortunately, due to

isomorphism  $SO(3, 2) \sim Sp(2, R)$ , there exists also more concise representation in terms of  $4 \times 4$  matrices. The symplectic connection reads

$$\mathcal{A} = \begin{pmatrix} C & D \\ F & -C^T \end{pmatrix}, \quad (30)$$

where  $D, F, C$  are  $2 \times 2$  matrices,  $D^T = D$ ,  $F^T = F$ ,

$$\begin{aligned} C &= \frac{1}{2} \left\{ \tau^{03} I_2 - \tau^{\theta 2} \sigma_x - i \tau^{12} \sigma_y + \tau^{\theta 1} \sigma_z \right\}, \\ D &= \frac{1}{2} \left\{ (\tau^{0\theta} - \tau^{\theta 3}) I_2 + (\tau^{23} - \tau^{02}) \sigma_x + (\tau^{01} - \tau^{13}) \sigma_z \right\}, \\ F &= \frac{1}{2} \left\{ -(\tau^{0\theta} + \tau^{\theta 3}) I_2 - (\tau^{23} + \tau^{02}) \sigma_x + (\tau^{01} + \tau^{13}) \sigma_z \right\}. \end{aligned} \quad (31)$$

Here  $I_2$  is a unit matrix and  $\sigma_x, \sigma_y, \sigma_z$  are Pauli matrices with  $\sigma_z$  diagonal. In view of (4), the equations of motion of the EMDA sigma-model are equivalent to vanishing of the curvature (5) related to (30). This implies the existence of the symmetric symplectic  $4 \times 4$  matrix  $g \in Sp(2, R)/U(2)$  entering Belinskii–Zakharov representation.

To summarize: we have shown that target spaces corresponding to the static EMD with an arbitrary dilaton coupling and the stationary EMDA systems in 4 dimensions are symmetric Riemannian spaces isomorphic to cosets  $SO(2, 1)/SO(2)$  and  $SO(3, 2)/(SO(3) \times SO(2))$  respectively (or, equivalently,  $SL(2, R)/U(1)$  resp.  $Sp(2, R)/U(2)$ ). This ensures zero-curvature representation of the equations of motion and existence of the Lax-pair in the axisymmetric case. The inverse scattering transform method can be applied to deal with both systems, in particular, to construct multisoliton solutions. Current algebras associated with  $SL(2, R)$  and  $Sp(2, R)$  generate infinite-dimensional GKC-type symmetries. Obviously, the whole reasoning can be generalized to the case of a space-like initial Killing vector field, as well as to the case of the Euclidean signature of the 4-space.

As it was noted in [10], the isometry group of the EMDA target space is larger than the product of well-known  $T$  and  $S$  string dualities [11]. Now it is clear that, on the class of space-times admitting a two-parameter Abelian isometry group, both these symmetries are particular elements of the infinite-dimensional GKC-type group. The implications of this to the *exact* string theory are still to be explored. An intriguing question is whether classical integrability of the 2-dim reduced EMDA system entails the possibility of explicit construction of new classes of exact string backgrounds in terms of the gauged WZW models. This issue will be discussed in a subsequent publication.

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